Quantum Darwinism in a Mixed Environment

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(Received 29 April 2009; revised manuscript received 8 August 2009; published 8 September 2009)

Quantum Darwinism recognizes that we—the observers—acquire our information about the ''systems of interest'' indirectly from their imprints on the environment. Here, we show that information about a system can be acquired from a mixed-state, or *hazy*, environment, but the storage capacity of an environment fragment is suppressed by its initial entropy. In the case of good decoherence, the mutual information between the system and the fragment is given solely by the fragment's entropy increase. For fairly mixed environments, this means a reduction by a factor $1-h$, where h is the *haziness* of the environment, i.e., the initial entropy of an environment qubit. Thus, even such hazy environments eventually reveal the state of the system, although now the intercepted environment fragment must be larger by $\sim(1-h)^{-1}$ to gain the same information about the system.

DOI: 10.1103/PhysRevLett.103.110402 PACS numbers: 03.65.Ta, 03.65.Yz

How the classical world arises from an ultimately quantum substrate has been a question since the advent of quantum mechanics [1–7]. Decoherence is now commonly used to study this quantum-classical transition [8–10]. Its theory, however, treats the environment as a sink where information about the system gets lost forever. Yet the information deposited in the environment can be intercepted, and it is our primary source of information about the Universe. Indeed, decohering interactions with the environment can amplify and store an impression of the system. Amplification was invoked already by Bohr [11] in the context of measurements. Early [12], as well as more recent [9,13,14], discussions of decoherence note the importance of redundancy, and provide an information-theoretic framework for how the environment acts as an amplifier and as a source of information about the ''system of interest'' [15–19].

Quantum Darwinism reflects this new focus on the environment as a communication channel [15–17]. When one receives a fragment of the environment by, for instance, intercepting with one's eyes a portion of photons that are scattered off a system of interest (e.g., the text of this Letter), one acquires information about it. Previous studies found that, with an initially pure environment, one can acquire information about the preferred observables of the system even from small environment fragments [17]. This explains the emergence of objectivity, as it allows many initially ignorant observers to independently obtain nearly complete information and reach consensus about the state of the system by intercepting different fragments of the environment. Classicality of states can now be quantified in terms of the redundancy of information transferred to and recorded by the environment. However, it is unclear how well one can accumulate information starting with a mixed, or *hazy*, environment, such as one at finite temperature. Yet the photon environment that is responsible for the vast majority of the information we gain has precisely such a hazy character. This Letter shows that even hazy environments will, in the end, communicate a very clear image.

We study how information flows into a hazy environment using the quantum mutual information, $I(\mathcal{S}{:}\mathcal{F}) = H_{\mathcal{S}}(t) + H_{\mathcal{F}}(t) - H_{\mathcal{S}\mathcal{F}}(t)$, between the system $\mathcal{S}$ and some fragment $\mathcal{F}$ of the environment $\mathcal{E}$. Here, $H(t)$ is the von Neumann entropy at time t of the subsystem specified in its subscript. We begin with some general considerations about information transfer from $\mathcal{S}$ to $\mathcal{F}$ due to a purely decohering Hamiltonian (i.e., a Hamiltonian, $\mathbf{H}_{\mathcal{S}\mathcal{E}}$, that commutes with the preferred pointer observable of $\mathcal{S}$ [9]). Under evolution generated by such a Hamiltonian, $\mathcal{S}$ alone, as well as $\mathcal{S}$ plus a small fragment $\mathcal{F}$ of $\mathcal{E}$, will eventually become effectively decohered when coupled to $\mathcal{E}/\mathcal{F}$, i.e., the rest of $\mathcal{E}$. In this case of ''good decoherence,'' the state of a qubit $\mathcal{S}$ will evolve as

$$\rho_{\mathcal{S}}(0) = \begin{pmatrix} s_{00} & s_{01} \\ s_{10} & s_{11} \end{pmatrix} \rightarrow \rho_{\mathcal{S}}(t) \approx \begin{pmatrix} s_{00} & 0 \\ 0 & s_{11} \end{pmatrix}, \qquad (1)$$

where the final $\rho_{\mathcal{S}}$ is diagonal in its pointer basis (with obvious generalization to larger system sizes). The system plus the fragment will therefore become

$$\rho_{\mathcal{S}\mathcal{F}}(t) \approx \begin{pmatrix} s_{00}\mathcal{U}_0\rho_{\mathcal{F}}(0)\mathcal{U}_0^\dagger & 0 \\ 0 & s_{11}\mathcal{U}_1\rho_{\mathcal{F}}(0)\mathcal{U}_1^\dagger \end{pmatrix}, \qquad (2)$$

where $\mathcal{U}_i$ is the evolution operator projected onto the ith pointer state of $\mathcal{S}$. This is because the remaining portion of the environment, $\mathcal{E}/\mathcal{F}$, suffices to decohere $\mathcal{S}$ plus $\mathcal{F}$ while preserving the pointer basis of $\mathcal{S}$. The entropy of the resulting state, Eq. (2), is thus identical to the entropy of the state $\rho_{\mathcal{S}}(t) \otimes \rho_{\mathcal{F}}(0)$. The mutual information between $\mathcal{S}$ and $\mathcal{F}$ then becomes

$$I(\mathcal{S}{:}\mathcal{F}) = H_{\mathcal{F}}(t) - H_{\mathcal{F}}(0) \qquad (3)$$

in this case of good decoherence. This formula reduces to $I(\mathcal{S}{:}\mathcal{F}) = H_{\mathcal{F}}(t)$ for initially pure $\mathcal{E}$ [20]. Equation (3) shows that information about $\mathcal{S}$ stored in an initially uncorrelated fragment $\mathcal{F}$ is represented solely by $\mathcal{F}$'s increase in entropy, which is due to the interaction with $\mathcal{S}$. Further, it shows how the capacity for $\mathcal{F}$ to store information is suppressed by its initial entropy, $H_{\mathcal{F}}(0)$. In effect, when the initial state of a fragment $\mathcal{F}$ is completely mixed, it has zero capacity for information as $I(\mathcal{S}{:}\mathcal{F}) = 0$ always. Equation (3) is a general result for purely decohering Hamiltonians in the case of good decoherence.

We will now see more explicitly what these results entail by studying a solvable model of a single spin interacting with an environment of ${}^\sharp\mathcal{E}$ spins according to the purely decohering Hamiltonian

$$\mathbf{H}_{\mathcal{S}\mathcal{E}} = \frac{1}{2}\sum_{k=1}^{{}^\sharp\mathcal{E}} \sigma_{\mathcal{S}}^z \sigma_k^z. \tag{4}$$

The advantage of this model is that not only is the evolution solvable but the entropy of $\mathcal{F}$ can be efficiently computed, enabling the investigation of large $\mathcal{E}$ (e.g., we have gone up to $\mathcal{F}$ with ${}^\sharp\mathcal{F} = 200$ spins, giving a Hilbert space dimension of 2^{200}) [21].

We consider here an $\mathcal{S}$ initially uncorrelated with a symmetric $\mathcal{E}$, i.e., $\rho_{\mathcal{S}\mathcal{E}}(0) = \rho_{\mathcal{S}}(0) \otimes \rho_r^{\otimes {}^\sharp\mathcal{E}}$, where

$$\rho_r = \begin{pmatrix} r_{00} & r_{01} \\ r_{10} & r_{11} \end{pmatrix} \tag{5}$$

in the σ^z basis. The haziness h is the preexisting entropy of an environment qubit,

$$h = -\mathrm{tr}(\rho_r \log_2 \rho_r). \tag{6}$$

It attains its maximum value of 1 bit when the qubit is completely mixed. Other factors, such as the dimension or alignment of the eigenstates of ρ_r with the basis singled out by $\mathbf{H}_{\mathcal{S}\mathcal{E}}$, affect the information acquired by $\mathcal{E}$, as we discuss elsewhere along with our numerical procedure [21].

We start by elucidating the concept of good decoherence. At time t, the reduced density matrix of $\mathcal{S}$ is

$$\rho_{\mathcal{S}}(t) = \begin{pmatrix} s_{00} & s_{01}\Lambda_{\mathcal{E}}(t) \\ s_{10}\Lambda_{\mathcal{E}}^\star(t) & s_{11} \end{pmatrix}, \tag{7}$$

where $\Lambda_{\mathcal{E}}(t) = \prod_{k\in\mathcal{E}} \Lambda_k(t)$ and $\Lambda_k(t) = \cos(t) + \imath(r_{11} - r_{00})\sin(t)$ for all k in our model. The decoherence factor $\Lambda_{\mathcal{E}}(t)$ represents how much $\mathcal{S}$ has been decohered by $\mathcal{E}$. The reduced density matrix $\rho_{\mathcal{S}\mathcal{F}}(t)$ has a similar structure, but the decoherence factor is $\Lambda_{\mathcal{E}/\mathcal{F}}(t)$, i.e., a product of ${}^\sharp\mathcal{E} - {}^\sharp\mathcal{F}$ instead of ${}^\sharp\mathcal{E}$ individual $\Lambda_k(t)$. For sufficiently small $\Lambda_k(t)$ and/or ${}^\sharp\mathcal{E} \gg {}^\sharp\mathcal{F}$, this will result in both $\rho_{\mathcal{S}}(t)$ and $\rho_{\mathcal{S}\mathcal{F}}(t)$ being nearly diagonal in the pointer basis of $\mathcal{S}$. This is "good decoherence." Moreover, $\rho_{\mathcal{S}\mathcal{F}}(t)$ can be diagonalized exactly for arbitrary conditions [22], giving

$$I(\mathcal{S}{:}\mathcal{F}) = [H_{\mathcal{F}}(t) - H_{\mathcal{F}}(0)] + [H(\kappa_{\mathcal{E}}(t)) - H(\kappa_{\mathcal{E}/\mathcal{F}}(t))], \tag{8}$$

where $H(x) = -x\log_2 x - (1-x)\log_2(1-x)$ and

$$\kappa_{\mathcal{A}}(t) = \frac{1}{2}\left(1 + \sqrt{(s_{11} - s_{00})^2 + 4|s_{01}|^2|\Lambda_{\mathcal{A}}(t)|^2}\right). \tag{9}$$

In our symmetric model, $H_{\mathcal{F}}(0) = {}^\sharp\mathcal{F} h$. Equation (8) demonstrates that $I(\mathcal{S}{:}\mathcal{F})$ is given exactly by the good decoherence expression, Eq. (3), plus a term giving the deviation from good decoherence. Essentially everywhere except for short t and $\mathcal{F} \simeq \mathcal{E}$, decoherence is good and the deviation term is nearly zero since both $\Lambda_{\mathcal{E}}(t)$ and $\Lambda_{\mathcal{E}/\mathcal{F}}(t)$ are almost zero. For our model system with $r_{00} = 1/2$, there is a time $t = \pi/2$ when $\Lambda_k(t) = 0$ and thus the condition for good decoherence is satisfied exactly except when ${}^\sharp\mathcal{F} = {}^\sharp\mathcal{E}$.

The evolution of $I(\mathcal{S}{:}\mathcal{F})$ for an initially pure $\mathcal{E}$ is shown in Fig. 1(a). At first, there are no correlations between $\mathcal{S}$ and $\mathcal{E}$, giving $I(\mathcal{S}{:}\mathcal{F}) = 0$. As $\mathcal{S}$ decoheres, however, information is transferred to $\mathcal{E}$. As a consequence, $I(\mathcal{S}{:}\mathcal{F})$ increases. This increase is initially steep, but the total missing information about $\mathcal{S}$ is limited by its entropy $H_{\mathcal{S}}$. Therefore, a plateau develops (the "classical plateau" [17]) as $I(\mathcal{S}{:}\mathcal{F})$ approaches $H_{\mathcal{S}}$. It is seen in Fig. 1 as the flat region of the mutual information plot. The level of the plateau occurs at $H_{\mathcal{S}} = H(s_{00})$. Thus, by intercepting just a few spins from $\mathcal{E}$ one can gain nearly all the information about $\mathcal{S}$. More precisely, the redundancy R_δ is the number of times the state of $\mathcal{S}$ can be deduced within an accuracy given by the *information deficit* δ, i.e.,

$$R_\delta = \frac{{}^\sharp\mathcal{E}}{{}^\sharp\mathcal{F}_\delta} = \frac{1}{f_\delta}, \tag{10}$$

where ${}^\sharp\mathcal{F}_\delta$ is the least number of spins needed to acquire a mutual information greater than $(1-\delta)H_{\mathcal{S}}$ and f_δ is the corresponding fraction of $\mathcal{E}$. For this model, as the time approaches $t = \pi/2$ any single spin from $\mathcal{E}$ has nearly all the information about $\mathcal{S}$ (i.e., δ is small for one environment spin). The redundancy in this case is simply ${}^\sharp\mathcal{E}$, the size of $\mathcal{E}$: many observers can each capture a single degree of freedom from $\mathcal{E}$ and gain the same information about $\mathcal{S}$. The existence of the plateau implies redundancy and demonstrates the validity of the quantum Darwinism paradigm. Such a plateau has been found starting with a pure $\mathcal{E}$ in other cases, including models with higher dimensional systems [16–18].

The key question we pose here is, what is the effect of starting with a hazy $\mathcal{E}$? Figure 1(b) plots $I(\mathcal{S}{:}\mathcal{F})$ versus t for $h \approx 0.8$. The figure shows that the classical plateau is still quite large and that it occurs at the same level, $H_{\mathcal{S}}$, as for an initially pure $\mathcal{E}$. After the plateau is reached, it stays flat until ${}^\sharp\mathcal{F} \approx {}^\sharp\mathcal{E}$. The plateau region will always develop for sufficiently large ${}^\sharp\mathcal{E}$ so long as $\mathcal{E}$ is not totally hazy ($h \neq 1$).

These findings are confirmed by an exact result for $I(\mathcal{S}{:}\mathcal{F})$ at $t = \pi/2$ and $r_{00} = 1/2$. In this case, $\rho_{\mathcal{F}}(\pi/2)$ can be diagonalized exactly [23]. The entropy of $\mathcal{F}$ is

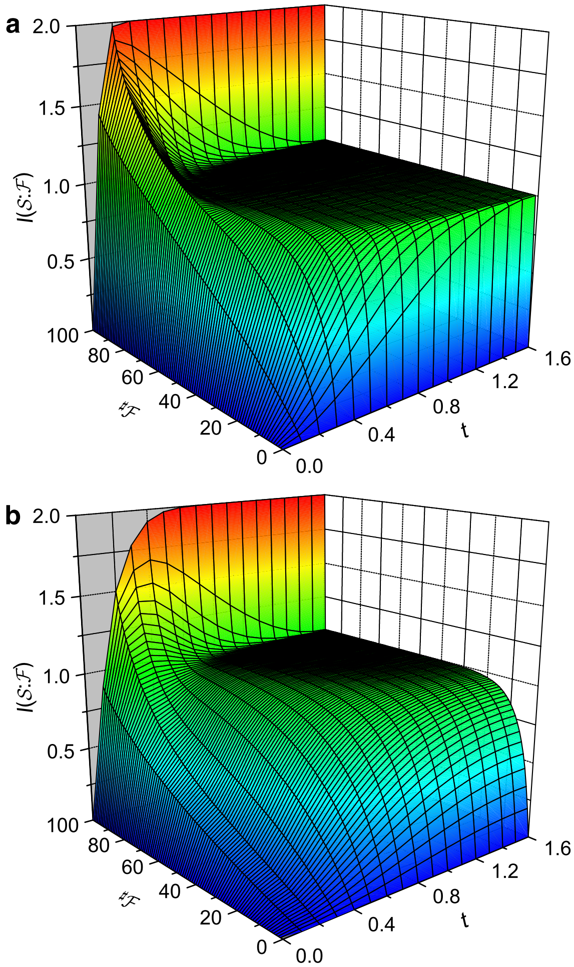

FIG. 1 (color online). Mutual information between $\mathcal{S}$ and a fragment $\mathcal{F}$ for $^{\#}\mathcal{E} = 100$, $\rho_{\mathcal{S}}(0) = |+\rangle\langle+|$, and $r_{00} = 1/2$, where $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$. An initially (a) pure $\mathcal{E}$, $h = 0$, and (b) hazy $\mathcal{E}$, $h \approx 0.8$. At $t = 0$, $\mathcal{S}$ and $\mathcal{E}$ are not correlated but, with time, correlations develop. As the system approaches good decoherence, a long plateau region forms where information is redundantly recorded in $\mathcal{E}$. This level occurs at the value of the entropy of $\mathcal{S}$ when decohered, which here is at $H_{\mathcal{S}} = 1$. For sufficiently large $^{\#}\mathcal{E}$, the mutual information will eventually reach H_S regardless of the initial haziness, except for a completely mixed initial $\mathcal{E}$, $h = 1$. However, the plateau is attained more slowly and only for larger fragments as $\mathcal{E}$ gets more mixed. When all of $\mathcal{E}$ is captured, the mutual information jumps to its maximum value of $2H_S$ (signifying complete quantum correlation of $\mathcal{E}$ with $\mathcal{S}$) so long as a well-defined plateau exists.

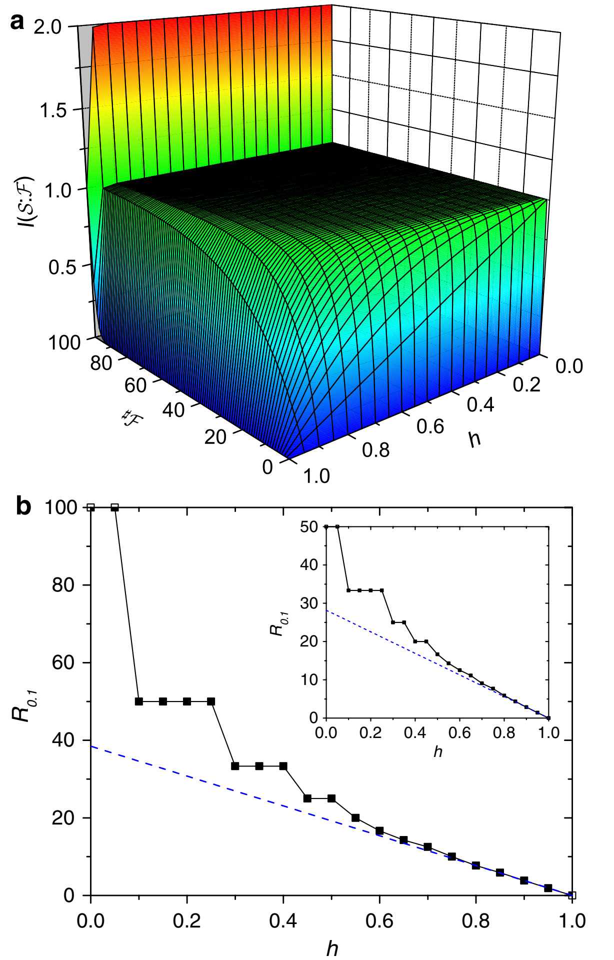

FIG. 2 (color online). (a) Mutual information at $t = \pi/2$ versus $^{\#}\mathcal{F}$ and h for the same conditions as in Fig. 1. (b) Redundancy versus h for the information deficit $\delta = 0.1$ and at $t = \pi/2$ (the inset shows $t = \pi/3$). The black line (squares) is the exact data. The redundancy can only take on rational values with $^{\#}\mathcal{E}$ in the numerator because of the discrete nature of the spin environment, which is particularly visible at high redundancy. The blue dashed line is that obtained by the scaling $1 - h$, which is a good approximation when h is near one. Thus, even initially mixed $\mathcal{E}$ can store information about $\mathcal{S}$ in many copies. However, it takes larger $^{\#}\mathcal{F}$ to acquire the same information about $\mathcal{S}$.

$$H_{\mathcal{F}}(\pi/2) = -\sum_{n=0}^{^{\#}\mathcal{F}} \binom{^{\#}\mathcal{F}}{n} \lambda_{\mathcal{F}}(n) \log_2[\lambda_{\mathcal{F}}(n)], \tag{11}$$

where n labels the degenerate eigenvalues $\lambda_{\mathcal{F}}(n) = s_{00}\lambda_-^n \lambda_+^{^{\#}\mathcal{F}-n} + s_{11}\lambda_-^{^{\#}\mathcal{F}-n}\lambda_+^n$ and $\lambda_\pm = 1/2 \pm |r_{01}|$ are the eigenvalues of ρ_r. Figure 2(a) shows $I(\mathcal{S}{:}\mathcal{F})$ versus $^{\#}\mathcal{F}$ and h at $t = \pi/2$. The plateau region is reached very rapidly except for h near 1, i.e., very near to a completely mixed $\mathcal{E}$. The redundancy for the information deficit $\delta = 0.1$ is plotted in Fig. 2(b) for $t = \pi/2$ and $t = \pi/3$. For h near 1, the redundancy scales as $R \propto 1 - h$: The information storage ability of $\mathcal{E}$ is suppressed by its initial entropy. This is also reflected in $I(\mathcal{S}{:}\mathcal{F})$ for very mixed states where $I(\mathcal{S}{:}\mathcal{F}) \approx (1 - h)^{\#}\mathcal{F}$, when $^{\#}\mathcal{F}$ is small and under the conditions in the figure.

These results indicate that a hazy $\mathcal{E}$ can be thought of as a noisy communication channel with degraded capacity, $1 - h$. The loss of the channel capacity can be depicted in terms of the overlap of two peaks in a bimodal probability distribution over subspaces of $\mathcal{F}$ with n identical records indicating a σ^y eigenstate and $^{\#}\mathcal{F} - n$ records indicating the orthogonal eigenstate. At $t = \pi/2$ the probability distribution is given by the two peaks

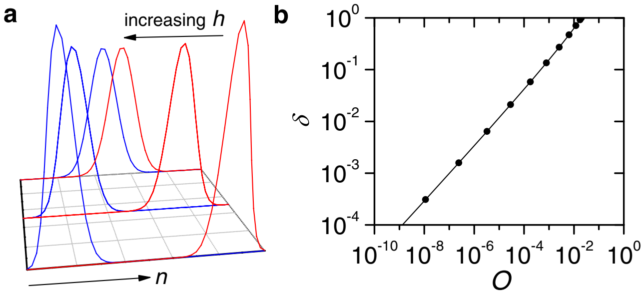

FIG. 3 (color online). (a) Bimodal probability distribution for the state of $\mathcal{F}$ to be in a subspace n, where $n = 0, \ldots, {}^{\sharp}\mathcal{F}$ indexes the number of identical records. The left (blue) peak, P_L, is correlated with $|0\rangle$ in $\mathcal{S}$ and the right (red), P_R, with $|1\rangle$. When $\mathcal{E}$ is initially nearly pure, $h \sim 0$, the two peaks are well separated. Upon increasing h, the two peaks start to overlap. (b) The information deficit δ versus the overlap $O = \sum_n P_L(n) P_R(n)$ for ${}^{\sharp}\mathcal{F} = 50$ and different h. A larger overlap is associated with a decreased capacity for information about $\mathcal{S}$, i.e., an increased information deficit. The parameters are the same as Fig. 1.

$$P_L(n) = s_{00} \binom{{}^{\sharp}\mathcal{F}}{n} \lambda_-^n \lambda_+^{{}^{\sharp}\mathcal{F}-n} \tag{12a}$$

and

$$P_R(n) = s_{11} \binom{{}^{\sharp}\mathcal{F}}{n} \lambda_-^{{}^{\sharp}\mathcal{F}-n} \lambda_+^n, \tag{12b}$$

with $n = 0, \ldots, {}^{\sharp}\mathcal{F}$. The bimodal structure is the result of information about $\mathcal{S}$ branching into sectors of $\mathcal{E}$'s Hilbert space. For initially very pure $\mathcal{E}$ and reasonably large ${}^{\sharp}\mathcal{F}$, these two sectors are distinct, allowing one to resolve the peaks, as shown in Fig. 3(a). In this case, $I(\mathcal{S}:\mathcal{F})$ is near its plateau value. As the peak overlap increases, the evidence about the state of $\mathcal{S}$ imprinted in $\mathcal{F}$ becomes less conclusive. Thus, a more hazy environment [Fig. 3(b)] or a smaller fragment result in an increased information deficit.

To conclude, we studied quantum Darwinism in the case of a hazy environment. For good decoherence and purely decohering Hamiltonians, we demonstrated that the mutual information acquired by some fragment of the environment is directly related to the entropy increase of that fragment. This shows that the capacity of the environment to accept information is suppressed by its initial entropy. Thus, a hazy environment acts like a noisy communication channel, transmitting all the information about the system, but at a lower rate. By examining a model system, we illustrated that, despite this diminished channel capacity, the region of redundant information storage is still reached for quite mixed environments. This work leads to questions related to recent research on representing environments in compact forms in order to accurately simulate a dynamical quantum system [24]: What role does the information acquired play in the representation of the environment, and, vice versa, does a compact representation yield the same redundancy as the full $\mathcal{E}$? Above all, however, our results verify the environment's capability to communicate information.

We would like to thank G. Smith, J. Yard, and M. Zubelewicz. This research is supported by the U.S. Department of Energy through the LANL/LDRD Program.

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[23] $\mathcal{F}$'s reduced density matrix is given by $s_{00} \otimes_{k\in\mathcal{F}} [\mathcal{V}(t)\rho_r \mathcal{V}(-t)] + s_{11} \otimes_{k\in\mathcal{F}} [\mathcal{V}(-t)\rho_r \mathcal{V}(t)]$. At $t = \pi/2$ and $r_{00} = 1/2$, the two terms are diagonal in the same basis but with their eigenvalues exchanged.
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